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Axioms of an Experimental System

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In this paper a system of axioms is presented to define the notion of an experimental system. The primary feature of these axioms is that they are based solely on the mathematical notion of a direct product decomposition of a set. Properties of experimental systems are then developed. This includes defining negation, implication, conjunction, and disjunction on the set $\mathfrak{Q}$ of all binary experiments of the system and showing that the resulting structure is a regular orthomodular poset. The theory of observables of experimental systems is also developed. Finally, the usual models of experiments from classical as well as quantum physics are shown to satisfy the axioms of an experimental system, and a mechanism to create new models of the axioms is given.

1. INTRODUCTION

In this paper an axiomatic theory of experimental systems is developed based on the notion of direct product decompositions. As with many axiomatic theories, ours grows from the desire to isolate a small number of essential features of a well-studied situation. There are two benefits to such an approach—one gains a clearer picture of the workings of the primary model, and useful generalizations often present themselves. The familiar situation abstracted by our axioms is that of experiments of a quantum mechanical system.

In the standard approach to quantum mechanics, one associates to each quantum system a Hilbert space $\mathcal{H}$. An experiment of the system, which is assumed to have finitely many mutually exclusive and exhaustive outcomes, is taken to be a finite sequence $P_1, \ldots, P_n$ of projection operators of $\mathcal{H}$. The exact requirements on this sequence is that the P_i be pairwise orthogonal and jointly span $\mathcal{H}$. It is well known that any vector $v \in \mathcal{H}$ can be expressed as

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$v = P_1 v + \ldots + P_n v$, a fact often expressed in physical terms by saying that v is a superposition of vectors corresponding to the various possible outcomes the experiment. Using this fact, it is easy to see that each experiment gives rise to a direct product decomposition $\mathcal{H} \cong \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$. It is this observation that lies at the heart of our axioms.

Given a bit of informality, the axioms defining an experimental system are easily described. First, it is assumed that one has an abstract set of experiments, and that each experiment is assigned a natural number representing its number of outcomes. It is further assumed that from any experiment e, other experiments can be built by combining various outcomes of e. If an experiment f can be built in this manner from an experiment e, one says that e refines f, and a set of experiments is called compatible in the case that a common experiment refines each. The first axiom asserts that a set S is associated with a system and that each n-ary experiment corresponds to an n-ary direct product decomposition $S \cong S_1 \times \cdots \times S_n$. The second axiom states that if an experiment e refines an experiment f, then the decomposition corresponding to e refines the decomposition corresponding to f. The final axiom asserts that if the decompositions corresponding to a set K of experiments is compatible, then the set K of experiments is compatible.

A heuristic comment may be worthwhile. The superposition principle of quantum mechanics asserts that a vector $v \in \mathcal{H}$ can be expressed as a sum $v = v_1 + \cdots + v_n$ of vectors corresponding to different possible outcomes of an experiment. In the axiomatic approach described above an analogous situation exists. Each element $s \in S$ is expressed as an ordered n-tuple $(s_1, \ldots, s_n)$ of elements corresponding to different possible outcomes of the experiment. In a sense, the fundamental observation is that an additive structure is not needed to describe a process of superposition.

While simple, a good deal can be proved from these axioms. As in the Hilbert space model, the set $\mathfrak{Q}$ of all binary experiments, or questions, plays an essential role. A unary operation of negation can easily be defined on $\mathfrak{Q}$ by setting *not e* to be the question formed by interchanging the order of the outcomes of e. Of more interest, a common refinement of compatible questions e, f can be used to construct questions *e and f* and *e or f*, and to define a relation *implies* on $\mathfrak{Q}$. Using the above axioms and a careful analysis of properties of direct product decompositions, one can show that ($\mathfrak{Q}$, *implies, not*) is a regular orthomodular poset with join and meet of compatible elements being given by the operations *and, or.*

The Boolean subalgebras of the structure $\mathfrak{Q}$ are used to develop the notion of observables of an experimental system. Here, too, decompositions of the set S play a fundamental role. Using the axioms of an experimental system, it is shown that the finite Boolean subalgebras of $\mathfrak{Q}$ with n atoms correspond exactly to the direct product decompositions of S associated with

n-ary experiments. The infinite Boolean subalgebras of $\mathfrak{Q}$ are also described in terms of decompositions of S. The correct notion here is not that of an infinite direct product, as one might first suspect, but rather of a continuously varying or Boolean product decomposition of S.

In the case that probabilities can be assigned to experiments in a regular fashion, these results on Boolean subalgebras can be put to use. One can show that each element $s \in S$ induces a finitely additive state φ_s: $\mathfrak{Q} \to [0, 1]$, and these states can in turn be used to associate numerical values to the expected value of an observable and to the probability of an observable yielding a value in a given Borel set of the reals.

The paper is organized in the following fashion. Section 2 contains the axioms defining an experimental system. Section 3 is devoted to logical aspects of the questions of a system. Section 4 introduces the axioms for experimental systems with probabilities. Section 5 pertains to observables. Section 6 presents various models of the axioms. These include mathematical models from classical physics, from the Hilbert space formulation of quantum mechanics, as well as a general method to create a wide range of new examples. Lengthy technical proofs have been removed from the main body of the paper and included in two appendices.

Finally, it must be remarked that this is a mathematical paper in which a mathematical notion of an experimental system is defined. No attempt is made here to give a precise physical definition of an experiment, and such is not required. Frequently, the names given to certain mathematical constructions will be physically suggestive, but it is not claimed that a complete physical interpretation has been given to the axioms introduced. It is not clear whether this difficult task has even been accomplished for the Hilbert space model of quantum mechanics from which these axioms are taken.

2. THE AXIOMS

While presenting formal mathematical definitions, it may be helpful to carry on an informal discussion as motivation. Our first remark is that an experiment e is to have finitely many mutually exclusive and exhaustive outcomes which are numbered outcome one through outcome n. The number of outcomes of an experiment e will be called the *arity* of the experiment. In abstracting certain properties of the set of all experiments we arrive at the following notion.

Definition 2.1. A leveled set consists of a nonempty set E, together with a map from E into the natural numbers. The natural number associated with an element $e \in E$ is called the *arity* of e.

Before proceeding further, it may be helpful to have an example of an experiment from the familiar quantum model. Consider the experiment which sends a particle through a certain type of magnetic device. Depending on whether the particle has spin $+1$, spin 0, or spin -1, the trajectory of the particle is sent along one of three paths. Detectors are placed at suitable spots along these three paths. If we agree to label the spin $+1$, 0, -1 outcomes as outcomes one, two, and three respectively, this determines a ternary experiment e. Of course, if one were to choose a different numbering system for the outcomes, say spin 0, -1, $+1$ corresponding to outcomes one, two, and three respectively, a closely related but different experiment would be obtained. As a convenient notational device, this new experiment is referred to as $(\{2\}, \{3\}, \{1\})e$.

A more interesting situation occurs when one considers experiments that can be built from the experiment e by combining various outcomes of e. Suppose the detectors placed on the spin 0 and spin -1 paths are removed, a device to merge the spin 0 and spin -1 paths is inserted, and a detector placed at a suitable spot on this new path. The result would have two possible outcomes. By assigning outcome one to the spin $+1$ detector and outcome two to the newly inserted detector, a new experiment is formed. As a notational convenience, this new experiment will be denoted $(\{1\}, \{2, 3\})e$. Our next task is to formalize some of the notation that will be required to abstract these notions.

Definition 2.2. An ordered partition of a natural number $n \geq 1$ is a finite sequence σ of pairwise disjoint subsets of $\{1, \ldots, n\}$ which cover $\{1, \ldots, n\}$. The number of terms in the sequence is denoted $\|\sigma\|$, and $\sigma(i)$ denotes the ith term of the sequence.

Of course, if one begins with an experiment e that has a relatively large number of outcomes, it is possible to create a new experiment σe by combining some of the outcomes of e. But the process need not stop here. By combining outcomes of this new experiment σe, one may produce yet another experiment, say $\phi(\sigma e)$. Clearly this final experiment is equivalent to one constructed directly from e. With a bit of foresight into the types of situations which will arise, the following extension is provided to our notation.

Definition 2.3. Let $\mathbb{O}_n$ denote the collection of all ordered partitions of n, and $\mathbb{O}$ denote the collection of all ordered partitions of natural numbers. Define a partial binary operation on $\mathbb{O}$ as follows: if σ is an ordered partition of n and ϕ is an ordered partition of $\|\sigma\|$, define $\phi\sigma$ to be the ordered partition of n with $(\phi\sigma)(i) = \cup\{\sigma(j): j \in \phi(i)\}$ for each $i \leq \|\phi\|$. Finally, for $n \geq 1$ define i_n to be the ordered partition $(\{1\}, \ldots, \{n\})$.

I am grateful to M. Gehrke for pointing out that ordered m-ary partitions of n correpond to functions from the set $\{1, \ldots, n\}$ to the set $\{1, \ldots, m\}$, an observation exploited in the proof of the following lemma.

Lemma 2.4. When defined, $\lambda(\phi\sigma) = (\lambda\phi)\sigma$.

Proof. For σ an m-ary partition of n, consider f_σ: $\{1, \ldots, n\} \to \{1, \ldots, m\}$, where $f_\sigma(i) = j$ iff $i \in \sigma(j)$. One easily checks $f_\phi \circ f_\sigma = f_{\phi\sigma}$. ■

The set $\mathbb{O}$ was introduced to describe a certain action on the set of experiments. It is of obvious importance to describe key features of this action.

Definition 2.5. An action of $\mathbb{O}$ on a leveled set E associates to each n-ary element $e \in E$ and each ordered partition σ of n a $\|\sigma\|$-ary element σe of E such that $(\phi\sigma)e = \phi(\sigma e)$ and $i_n e = e$.

Certain leveled sets acted on by $\mathbb{O}$ will be the basic objects of study. A few suggestive terms presented in the following definition will make discussions a bit easier.

Definition 2.6. Let E be a leveled set acted on by $\mathbb{O}$ and e, f be elements of E. Then f is said to be built from e if there is σ with σe defined and equal to f. A subset $K \subseteq E$ is called compatible if for each finite subset $K' \subseteq K$ there is a single member of E from which each member of K' can be built.

With the basic notions in hand, the task of relating direct product decompositions to experiments can begin. The first step is to give a precise definition of the term.

Definition 2.7. An n-ary decomposition of a set S consists of a sequence $S_1, \ldots, S_n$ of sets and an isomorphism $f: S \to S_1 \times \cdots \times S_n$. This isomorphism is often written $S \cong_f S_1 \times \cdots \times S_n$. From any such decomposition one obtains a sequence of maps $f_1, \ldots, f_n$, with $f_i: S \to S_i$, such that $f(s) = (f_1(s), \ldots, f_n(s))$.

A few difficulties immediately present themselves. Consider, for example, the case that S is a finite set with a prime number of elements p. In a sense, there are two different ways to decompose S as a direct product of two sets—as a p-element set times a one-element set, or as a one-element set times a p-element set (the order of the factors is important!). However, there are infinitely many different p-element sets, hence infinitely many binary direct product decompositions of S. To capture the idea that it is the method in which S is decomposed, not the actual sets used in the decomposition, the following definition is employed.

Definition 2.8. An equivalence relation is defined on the class of all decompositions of a set S by setting $S \cong_f S_1 \times \cdots \times S_m$ equivalent to $S \cong_g T_1 \times \cdots \times T_n$ if $m = n$ and for each $i \leq n$ there is an isomorphism h_i: S_i

$\rightarrow T_i$ with $h_i \circ f_i = g_i$. The notation $[S \cong_f S_1 \times \cdots \times S_n]$ is used for the equivalence class of the given decomposition, and $\mathfrak{D}(S)$ denotes the collection of all equivalence classes of decompositions of S.

The set $\mathfrak{D}(S)$ clearly forms a leveled set where the natural number associated with an equivalence class of decompositions $[S \cong S_1 \times \cdots \times S_n]$ is n. The following result provides a first link between experiments and decompositions. Its proof is not difficult, but rather lengthy, and is left to the reader.

Lemma 2.9. An action of $\mathbb{O}$ may be defined on the leveled set $\mathfrak{D}(S)$ by setting $\sigma[S \cong_f S_1 \times \cdots \times S_n]$ to be the equivalence class of the obvious decomposition $S \cong T_1 \times \cdots \times T_{\|\sigma\|}$ where $T_i = \prod\{S_j\colon j \in \sigma(i)\}$.

A few simple examples may help to untangle the notation. Given a decomposition $S \cong S_1 \times \cdots \times S_4$ and an ordered partition $\sigma = (\{2\}, \{1, 4\}, \{3\})$, it follows that

$$\sigma[S \cong S_1 \times S_2 \times S_3 \times S_4] = [S \cong S_2 \times (S_1 \times S_4) \times S_3]$$

It is perhaps worthwhile to point out that the empty set may occur in an ordered partition σ of n. This poses no difficulties if one remembers that the union of the empty family of sets is the empty set and that the product of the empty family of sets is a one-element set. For instance if $\chi = (\emptyset, \{1\ 3\}, \{2, 4\})$, then

$$\chi[S \cong S_1 \times S_2 \times S_3 \times S_4] = [S \cong \{\circ\} \times (S_1 \times S_3) \times (S_2 \times S_4)]$$

Here $\{\circ\}$ is used to denote any one-element set. Informally, if e is an experiment with four outcomes, and $\chi = (\emptyset, \{1, 3\}, \{2, 4\})$, one interprets χe as an experiment with three outcomes—the second outcome of χe is built by combining the first and third outcomes of e, the third outcome of χe is built by combining the second and fourth outcomes of e, and the first outcome of χe never happens. The preliminaries set, it is now possible to define the notion of an experimental system.

Definition 2.10. An experimental system consists of a leveled set E which is acted on by $\mathbb{O}$, a set S, and an embedding $D{:}E \rightarrow \mathfrak{D}(S)$ which satisfies the three axioms below. For convenience, elements of E are called experiments.

Axiom 1. If e is an n-ary experiment, then De is n-ary.

Axiom 2. $D(\sigma e) = \sigma(De)$ for each n-ary experiment e and each $\sigma \in \mathbb{O}_n$.

Axiom 3. For a set K of binary experiments, $D[K]$ compatible implies K compatible.

The effects of weakening Axiom 3 to apply only to sets of two binary experiments, and strengthening Axiom 3 to apply to arbitrary sets of experiments, will also be studied. Roughly, the weakening has little overall effect, and the apparently stronger version is already a consequence of the axioms presented above.

3. THE LOGICAL STRUCTURE OF QUESTIONS

Let $\mathfrak{Q}$ be the set of all binary experiments of an experimental system. Such binary experiments are often called *Yes–No* questions, or simply questions. Outcome one of a question is referred to as the *Yes* outcome, and outcome two as the *No* outcome. The aim is to define questions *True* and *False*, to define the negation of a question, and for compatible pairs of questions, to define their conjunction, disjunction, and the relationship of implication.

Lemma 3.1. There are unique questions *True*, *False* with $D(True) = [S \cong S \times \{\circ\}]$ and $D(False) = [S \cong \{\circ\} \times S]$. Here $\{\circ\}$ is a one-element set.

Proof. Unicity follows as D is an embedding. For existence, the collection of experiments of an experimental system is by definition nonempty. If e is an n-ary question, set $True = (\{1, \ldots, n\}, \emptyset)e$ and $False = (\emptyset, \{1, \ldots, n\})e$. ■

A proof of the following result is found in appendix A.

Lemma 3.2. If e, f are compatible questions, then there is a unique experiment g with four outcomes such that $e = (\{1, 2\}, \{3, 4\})g$ and $f = (\{1, 3\}, \{2, 4\})g$. The experiment g is called the standard refinement of the ordered pair (e, f).

Lemma 3.3. Let g be an n-ary experiment and σ, ϕ be ordered partitions of n with $\|\sigma\| = \|\phi\| = 2$. Then the standard refinement of the ordered pair of questions $(\sigma g, \phi g)$ is μg, where $\mu = (\sigma(1) \cap \phi(1), \sigma(1) \cap \phi(2), \sigma(2) \cap \phi(1), \sigma(2) \cap \phi(2))$.

Proof. As $\|\phi\| = 2$, the sets $\phi(1)$ and $\phi(2)$ form a partition of the set $\{1, \ldots, n\}$. Therefore $\mu(1) \cup \mu(2) = \sigma(1)$. Using this and other similar equations, it follows that $(\{1, 2\}, \{3, 4\})\mu = \sigma$ and $(\{1, 3\}, \{2, 4\})\mu = \phi$. ■

The above result on the existence and uniqueness of standard refinements allow for the definition of the logical operations desired.

Definition 3.4. Let e and f be compatible questions, and g be the standard refinement of (e, f). Then define the following questions:

(1) $(e \text{ and } f) = (\{1\},\{2,3,4\})g$.
(2) $(e \text{ or } f) = (\{1,2,3\}, \{4\})g$.
(3) $(\text{not } e) = (\{2\}, \{1\})e$.

A relation *implies* is then defined on $\mathfrak{Q}$ by setting e *implies* f if e, f are compatible and ((not e) or f) $=$ *True*.

A comment on the interpretation of these operations is in order. The compatibility of questions e, f means that there is an experiment g with four outcomes such that the *Yes* outcome of e corresponds to the combination of the first two outcomes of g, the *No* outcome of e corresponds to the combination of the last two outcomes of g, the *Yes* outcome of f corresponds to the combination of the first and third outcomes of g, and the *No* outcome of f corresponds to the combination of the second and fourth outcomes of g. The experiment (e *and* f) described above does not involve actually conducting either experiment e or experiment f. Instead, one starts with the experiment g, leaves the first outcome of g alone, and combines the final three outcomes of g. A *Yes* outcome to (e *and* f) then corresponds to the first outcome of g and a *No* outcome to (e *and* f) corresponds to the combination of the final three outcomes of g.

One may or may not believe that this process gives a reasonable definition of the conjunction of two experiments. However, it certainly does give an operation on the collection of all binary experiments of a system. This operation, and the corresponding operation for disjunction, can then be studied under whatever names one should choose. It is the task here to show the algebraic character of these operations has much in common with that of their namesakes from classical logic. A first step toward this goal is provided by the following.

Proposition 3.5. For an experiment g, any two questions of $B = \{\sigma g\colon \|\sigma\| = 2\}$ are compatible. Moreover, B is closed under the operations *and*, *or*, *not*, *True*, *False*, and forms a Boolean algebra under these operations with the partial ordering given by *implies*.

Proof. By definition, any two questions in B are compatible, and it follows from Lemma 3.3 that B is closed under the logical operations. To show B is Boolean, it is enough to show the map φ from the power set of $\{1, \ldots, n\}$ onto B defined by $\varphi(A) = (A, A^c)g$ is a homomorphism. Note that

$$\textit{not}\ \varphi(A) = \textit{not}((A, A^c)g) = (A^c, A)g = \varphi(A^c)$$

Also, using the description of standard refinements given in Lemma 3.3, we have

$$\varphi(A)\ \textit{and}\ \varphi(C) = (A \cap C, A^c \cup C^c)g = \varphi(A \cap C)$$

Clearly a similar result holds for conjunctions as well, and $\varphi(\emptyset) = \textit{False}$. Thus φ is a homomorphism. ∎

Therefore, conjunction, disjunction, negation, and implication behave classically when restricted to certain small subsets of the set $\mathfrak{Q}$ of questions. Next, the behavior of these notions on $\mathfrak{Q}$ as a whole is examined. From the outset, the possibility that certain experiments are not compatible has been acknowledged. Thus, there is no reason to expect, or desire, notions of conjunction and disjunction valid for all pairs of questions. Still, there is much to ask of the system. For example, one would like implication to be transitive, and if all of the operations in the following expression are known to be defined, one would like to conclude

$$e \textit{ and } (f \textit{ or } g) = (e \textit{ and } f) \textit{ or } (e \textit{ and } g) \tag{3.1}$$

This and much more is true of the questions of any experimental system. Before demonstrating this, it is useful to introduce some terminology.

Definition 3.6. A subset Y of an orthomodular poset X is a subalgebra of X if Y is closed under orthocomplementation and contains the least upper bound $x \oplus y$ of any two orthogonal elements belonging to Y. Clearly a subalgebra of an orthomodular poset is itself an orthomodular poset. If a subalgebra happens to be a Boolean algebra, it is called a Boolean subalgebra.

A proof of the following well known result can be found in ref. 4.

Lemma 3.7. If B is a Boolean subalgebra of an orthomodular poset X, then any two elements of B have a least upper bound in X which agrees with their least upper bound in B. Similar remarks hold for greatest lower bounds.

Much of the work in establishing the following theorem is done in refs. 3 and 4, and the remaining details are provided in Appendix A.

Theorem 3.8. (1) ($\mathfrak{Q}$, *implies*, *not*, *False*, *True*) is an orthomodular poset. (2) A set K of questions is contained in a Boolean subalgebra of $\mathfrak{Q}$ iff K is compatible, which occurs iff any two questions in K are compatible. (3) The operations in any Boolean subalgebra of $\mathfrak{Q}$ are given by *and*, *or*, *not*, *False*, *True*. (4) If B is a finite Boolean subalgebra of $\mathfrak{Q}$ with n atoms, then there exists an n-ary experiment g such that $B = \{\sigma g \colon \|\sigma\| = 2\}$.

This theorem is much more than sufficient to provide condition (3.1) above. Indeed, for all the terms in (3.1) to be defined, it is necessary that any two of e, f, g are compatible. Then all three questions are compatible! and are elements of a Boolean subalgebra B of $\mathfrak{Q}$. As the operations in any Boolean subalgebra of $\mathfrak{Q}$ are given by *and*, *or*, *not*, condition (3.1) follows immediately. The following results clarify the consequences of altering the third axiom of an experimental system. Their proofs are found in Appendix A.

Proposition 3.9. Let $\mathfrak{Q}$ be the questions of a system which satisfies Axioms 1 and 2, and a weakened version of Axiom 3 required to hold only for sets K consisting of two binary experiments. Then (1) ($\mathfrak{Q}$, *implies*, *not*, *False*, *True*) is an orthomodular poset. (2) A set K of questions is contained in a Boolean subalgebra of $\mathfrak{Q}$ iff any two questions in K are compatible. (3) The operations in any Boolean subalgebra of $\mathfrak{Q}$ are given by *and*, *or*, *not*, *False*, *True*.

Proposition 3.10. There is a system which satisfies Axioms 1 and 2, and a weakened version of Axiom 3 required to hold only for sets K consisting of two binary experiments, but does not satisfy Axiom 3.

Proposition 3.11. For an experimental system, Axiom 3 holds for any set K of experiments, binary or otherwise.

4. PROBABILITIES

In this section additional features are incorporated into the notion of an experimental system to allow numerical values to be associated with experiments in a regular fashion.

Definition 4.1. A map $p: S \to [0, 1]^n$ is called an n-ary probability map on S if $\Sigma_1^n\, p_i(s)$ is either 0 or 1 for each $s \in S$. $\mathcal{P}(S)$ denotes the collection of all probability maps on S.

$\mathcal{P}(S)$ naturally forms a leveled set with the natural number associated to an element $p: S \to [0, 1]^n$ being n. The following result is easy to verify and its proof left to the reader.

Lemma 4.2. An action of $\mathcal{O}$ on $\mathcal{P}(S)$ may be defined by setting σp to be the $\|\sigma\|$-ary probability map with $(\sigma p)_i(s) = \Sigma\{p_j(s): j \in \sigma(i)\}$.

A simple example may help ease the notation. A probability map $p: S \to [0, 1]^4$ is most conveniently written using its component maps as $(p_1, \ldots, p_4)$. This means that $p(s)$ is the element $(p_1(s), \ldots, p_4(s))$ of $[0, 1]^4$ for each s in S. For the ordered partition $\sigma = (\{2\}, \{1, 4\}, \{3\})$, one has $\sigma p = (p_2, p_1 + p_4, p_3)$, where the notation $p_1 + p_4$ denotes the usual sum of real-valued functions. The preliminaries aside, the required notion of an experimental system with probabilities is given.

Definition 4.3. An experimental system with probabilities consists of an experimental system $D: E \to \mathfrak{D}(S)$ together with a map $P: E \to \mathcal{P}(S)$ which satisfies the following axioms:

Axiom 4. If e is an n-ary experiment, then Pe is an n-ary probability map.

Axiom 5. $P(\sigma e) = \sigma(Pe)$ for each n-ary experiment e and each $\sigma \in \mathcal{O}_n$.

Note that D must be an embedding, but P need not. Often, elements of S are called pure states, or simply states, and $(Pe)_i(s)$ is called the probability of obtaining the ith outcome of experiment e, given the system is in state s.

Lemma 4.4. If e is an n-ary experiment and s is a state, then $\Sigma_1^n \, (Pe)_i(s) = 0$ if and only if $P(True)_1(s) = 0$.

Proof. Let $\sigma = (\{1, \ldots, n\}, \emptyset)$. Then $D(\sigma e) = \sigma(De) = [S \cong S \times \{\circ\}]$, so $\sigma e = True$. But $\Sigma_1^n \, (Pe)_i(s) = (\sigma(Pe))_1(s) = (P(\sigma e))_1(s) = P(True)_1(s)$. ■

States $s \in S$ for which $P(True)_1(s) = 0$ are called null states. From the above, a state s is a null state if and only if there is zero probability of obtaining any outcome of any experiment when the system is in state s.

5. OBSERVABLES

Informally, terms such as position and momentum are abstract notions used to discuss certain families of compatible experiments. These are customarily called observable quantities, and the particular manner in which numerical values are associated with an observable quantity is called its scaling. Here, notions of observable quantities and scalings are defined for experimental systems. Throughout, $\mathfrak{R}$ denotes the real numbers, and $[-\infty, \infty]$ the extended reals.

Definition 5.1. An observable quantity is a Boolean subalgebra B of the questions $\mathfrak{Q}$ of the system.

Recall that for any Boolean algebra B, there is a compact Hausdorff space Z which has a basis of sets which are both open and closed (clopen) such that B is isomorphic to the collection of all clopen subsets of Z. This space Z is called the Stone space of B. Elements of Z are maximal filters of B, and the clopen subsets of Z are exactly the ones of the form $c^* = \{F \in Z\colon c \in F\}$, where $c \in B$. In addition to the topology on Z, there is a σ-algebra of subsets of Z generated by the clopen sets. Measures and measurable functions on Z are understood to be with respect to this σ-algebra.

Definition 5.2. A scaling of an observable quantity is a real random variable on the Stone space of B, or in other words, a measurable map from the Stone space of B to the extended reals.

To assign probabilities to the outcomes of an experiment, the notion of an experimental system with probabilities was introduced. It is this same notion that is used to assign probabilities to measurements of an observable quantity. For the remainder, assume the experimental system has probabilities in the sense of the previous section, and for each state s define ψ_s: $\mathfrak{Q} \to [0, 1]$ by setting $\psi_s(e) = (Pe)_1(s)$.

Lemma 5.3. For an observable quantity B and a state s which is not null, the map ψ_s: $B \to [0, 1]$ is a finitely additive measure.

Proof. By Theorem 3.8 the operations on any Boolean subalgebra of $\mathfrak{Q}$ are given by *and*, *or*, *not*, *False*, *True*. Using the definition of null state, it follows that $\psi_s(True) = 1$. It remains only to show additivity for disjoint elements. Suppose $e, f \in B$ with $(e \text{ and } f) = False$, and let g be the standard refinement of the ordered pair (e, f) given by Lemma 3.2. As $e = (\{1, 2\}, \{3, 4\})g$, it follows that $(Pe)_1 = (Pg)_1 + (Pg)_2$, and similarly as $f = (\{1, 3\}, \{2, 4\})g$, it follows that $(Pf)_1 = (Pg)_1 + (Pg)_3$. From the definition of *and* given in Definition 3.4, it follows that $False = (e \text{ and } f) = (\{1\}, \{2, 3, 4\})g$, hence $0 = P(False)_1 = (Pg)_1$. From the definition of *or*, it follows that $P(e \text{ or } f)_1 = P((\{1, 2, 3\}, \{4\})g)_1 = (Pg)_1 + (Pg)_2 + (Pg)_3 = (Pe)_1 + (Pf)_1$. Therefore $\psi_s(e \text{ or } f) = \psi_s(e) + \psi_s(f)$. ■

The above shows that for each $s \in S$ which is not null, the map ψ_s: $Q \to [0, 1\}$ is a finitely additive state in the sense usually used when studying orthomodular posets.[8]

Proposition 5.4 Let B be an observable quantity. Then for each state s which is not null there is a unique probability measure μ_s on the Stone space of B with $\mu_s(e^*) = \psi_s(e)$ for each $e \in B$.

Proof. Consider the map ψ_s^* from the clopen sets of the Stone space of B to $[0, 1]$ defined by $\psi_s^*(e^*) = \psi_s(e)$. This map is finitely additive, and as Z is compact, it follows that ψ_s^* is a probability measure on the clopen subsets of Z in the sense of (ref 6, p. 10). So ψ_s^* has a unique extension to a probability measure μ_s on the σ-algebra of subsets of Z generated by the clopen subsets of Z (ref. 6, p. 23). ■

With the preliminaries aside, the key notions can now be defined.

Definition 5.5. Let f be a scaling of an observable quantity B and s be a state that is not null. The value $\mu_s(f^{-1}U)$ is called the probability that a measurement of B will yield a value in the Borel set U, under the scaling f, given the system is in state s. The expected value of the observable quantity B under the scaling f when the system is in state s is defined simply to be the expected value $\int_Z f \, d\mu_s$ of the scaling f with respect to the measure μ_s. Note also that a calculus of scalings is easily developed. For any measurable map φ on the extended reals, and any scaling f, define $\varphi(f)$ to be the scaling $\varphi \circ f$.

A simple example follows.

Example 5.6. Consider the observable quantity $B = \{True, False, e, not\ e\}$. Note that B has precisely two maximal filters, $e\!\uparrow$ and $not\ e\!\uparrow$, hence the Stone space Z is a two-element discrete space $\{e\!\uparrow, not\ e\!\uparrow\}$. For a state s which is not null, the measure μ_s is given by $\mu_s(\{e\!\uparrow\}) = \psi_s(e) = (Pe)_1(s)$, the probability of obtaining a *Yes* outcome to e when the system is in state s, and $\mu_s(\{not\ e\!\uparrow\})$ is the probability of obtaining a *No* outcome to e when the system is in state s. The scaling $f(e\!\uparrow) = 1.2$ and $f(not\ e\!\uparrow) = 1.7$ associates the numerical value 1.2 to a *Yes* outcome of e and 1.7 to a *No* outcome of e. The expected value when the system is in state s is $\int_Z f d\mu_s = 1.2(Pe)_1(s) + 1.7(Pe)_2(s)$.

While scalings of observable quantities are plentiful, one particular method of constructing scalings is likely to be of interest, especially to those familiar with the treatment of observables given in quantum logic. The situation is described by the following result.

Proposition 5.7. Let φ be a Boolean algebra homomorphism from the Borel subsets of the reals to an observable quantity B. Then the map f from the Stone space of B to the extended reals defined by $f(F) = \inf\{\lambda \in \Re: \varphi(-\infty, \lambda] \in F\}$ is both continuous and measurable.

Proof. It suffices to show that the inverse image under f of a basic open subset of the extended reals is a countable union of clopen sets. Let Q^+ be the set of positive rational numbers. Then

$$f^{-1}(a, \infty] = \{F: \varphi(-\infty, a + \epsilon] \notin F \text{ for some } \epsilon \in Q^+\}$$

hence $f^{-1}(a, \infty] = \cup_{Q^+} \varphi(a + \epsilon, \infty)^*$. Similarly, $f^{-1}[-\infty, b) = \cup_{Q^+}\varphi(-\infty, b - \epsilon)^*$. Thus f is both continuous and measurable. ■

To conclude this section, one important task remains—to show the fundamental connection between decompositions of S and observables. Recall that each n-ary experiment e gives rise to an n-ary direct product decomposition $S \cong S_1 \times \cdots \times S_n$ of the state space S. One may obviously consider the set $\{1, \ldots, n\}$ to be an indexing set of the factors used in this decomposition. For each state s, one then obtains a weighting w_s of the points $\{1, \ldots, n\}$ where the weight associated with the point i is $P(e)_i(s)$. A scaling then assigns to each outcome i a numerical value $f(i)$.

Informally, one might think of an observable as a type of limiting process of a system of ever finer finitary experiments. What then might correspond to the limit of the associated system of ever finer finitary direct product decompositions of S? To answer this, one requires the notion of a continuously varying, or Boolean product, decomposition of S. Roughly, this is a representation of S as a subalgebra of a direct product $\prod_Z S_z$ where the set used to index the factors is the Stone space Z. The situation then parallels the finitary case. An element of Z corresponds not to an outcome of an experiment, but to the limit of a system of ever finer outcomes. For a state s, one again obtains a weighting of the points of the indexing set Z. However, this weighting is no longer a simple point charge, but a measure μ_s. Finally, a scaling f assigns to each idealized outcome of Z a numerical value.

The heuristics aside, the exact nature of a Boolean product decomposition must be made precise, and the correspondence between these Boolean product decompositions and observable quantities described. The key to this is the notion of a sheaf.

Definition 5.8. A sheaf is an indexed family $(T_x)_{x \in X}$ of pairwise disjoint sets T_x, called stalks, together with topologies on $T = \cup_X T_x$ and on the indexing set X, such that each $t \in T$ has a neighborhood on which the natural projection π: $T \to X$ restricts to a homeomorphism. Given an open set $U \subseteq X$, define $\Gamma(U)$ to be the set of all continuous maps f: $U \to T$ with $f(x) \in \mathrm{T}_x$ for each $x \in U$. The elements of $\Gamma(U)$ are called sections over U.

By definition, $\Pi_X T_x = \{f\colon X \to T\colon f(x) \in T_x\}$ for any indexed family of sets. So for a sheaf $(T_x)_X$, the set $\Gamma(X)$ is clearly a subset of the direct product consisting of those choice functions which are continuously varying. In the special case that the base space Z of a sheaf $(T_z)_Z$ is the Stone space of a Boolean algebra, the sheaf is called a Boolean sheaf, and the set $\Gamma(Z)$ is called the Boolean product of $(T_z)_Z$. See ref. 1 for access to the literature on Boolean products and sheaves. The following result has a long history. Modulo the discussion of $\mathfrak{Q}$ provided by Theorem 3.8, it can reasonably be attributed to Pierce.

Theorem 5.9. Let B be an observable quantity and Z be its Stone space. Then there is a sheaf $(T_z)_Z$ such that $De = [S \cong \Gamma(e^*) \times \Gamma(not\ e^*)]$ for each $e \in B$.

As a final remark, it should be noted that many modifications of the notion of an observable quantity and scaling might be made. For example, it would be most reasonable to require $f^{-1}[\{\pm\infty\}]$ be of μ_s measure zero for each state s. Also, one might consider the possibility of requiring a scaling to be not only measurable, but continuous as well, or demanding that an observable quantity be a complete Boolean algebra. When one specializes to the Hilbert space model of quantum mechanics, all these stronger conditions are valid for the self-adjoint operators used to model observables. The aim here is to find the bare minimum one might require of an experimental system; one can specialize from there.

6. MODELS

In this section, various models of the axioms are presented. We begin with the Hilbert space model of quantum mechanics on which our axioms are based.

6.1. The Hilbert Space Model

Definition 6.1.1. Let $\mathcal{H}$ be a Hilbert space. Define E be the collection of all sequences $P_1, \ldots, P_n$ of projection operators whose ranges are pairwise orthogonal subspaces which together span $\mathcal{H}$, and define an action of $\mathbb{C}$ on E by setting the ith element of the sequence $\sigma(P_1, \ldots, P_n)$ to be $\Sigma\{P_j: j \in \sigma(i)\}$.

Given an experiment $e = (P_1, \ldots, P_n)$, each vector v is uniquely determined by the sequence $(P_1 v, \ldots, P_n v)$. It follows that $\mathcal{H}$ can be decomposed as the product $P_1(\mathcal{H}) \times \cdots \times P_n(\mathcal{H})$ of the ranges of these projection operators. Further, as these ranges are pairwise orthogonal, $\|v\|^2 = \|P_1 v\|^2 + \cdots + \|P_n v\|^2$. These facts allow the following definitions.

Definition 6.1.2. For $e = (P_1, \ldots, P_n)$, define De to be the equivalence class of the decomposition $\mathcal{H} \cong P_1(\mathcal{H}) \times \cdots \times P_n(\mathcal{H})$. Further, let Pe: $\mathcal{H} \to [0, 1]^n$ be the n-ary probability map whose ith component is given by $(Pe)_i(v) = \|P_i v\|^2/\|v\|^2$ for $v \neq 0$ and $(Pe)_i(v) = 0$ for $v = 0$.

The proof of the following result is found in Appendix B.

Proposition 6.1.3. The maps D: $E \to \mathcal{D}(\mathcal{H})$ and P: $E \to \mathcal{P}(\mathcal{H})$ form an experimental system with probabilities. Further, $(P, P^{\perp}) \rightsquigarrow P$ is an isomorphism between the orthomodular poset $\mathfrak{Q}$ of questions of the system and $Proj(\mathcal{H})$.

This shows that an experimental system with probabilities can be built from a Hilbert space. Moreover, the treatment of experiments and their probabilities in the usual Hilbert space model of quantum mechanics agrees exactly with the treatment given here. However, there is a crucial detail remaining. The treatment of observables given in the Hilbert space model of quantum mechanics appears very different from the notion of an observable of an experimental system. Roughly speaking, observables in the Hilbert space model correspond to self-adjoint operators on the Hilbert space $\mathcal{H}$. Our next task is to reconcile these two approaches. We begin by briefly describing how observables are treated in the usual Hilbert space model (see ref. 7 for a complete account).

Definition 6.1.4. A spectral measure is a map E: $\mathcal{B} \to Proj(\mathcal{H})$ from the Borel sets $\mathcal{B}$ of the reals $\mathfrak{R}$ to the projection operators of a Hilbert space with (1) $E(\Re) = 1$ and (2) $E(\cup_1^\infty S_n) = \Sigma_1^\infty E(S_n)$ for any sequence $S_1, S_2, \ldots$ of pairwise disjoint sets.

The key point is that to any self-adjoint operator A is associated a unique spectral measure E having certain properties (ref. 7, Theorems 5.6 and 6.3). We shall refer to this as the spectral measure associated with A. The following proposition is a special case of a more general result (ref. 7, Theorem 5.4).

Proposition 6.1.5. If E: $\mathcal{B} \to Proj(\mathcal{H})$ is a spectral measure, then for any nonzero $v \in \mathcal{H}$ the map ν_v: $\mathcal{B} \to [0, 1]$ defined using the inner product by setting $\nu_v(Y) = (v, E(Y)v)/\|v\|^2$ is a probability measure on the Borel sets $\mathcal{B}$.

With these facts about Hilbert spaces in hand, the treatment of observables in the usual Hilbert space model of quantum mechanics can be described. In this model, observables are associated with self-adjoint operators of the Hilbert space $\mathcal{H}$. Suppose A is the self-adjoint operator associated with a given observable and E is the spectral measure associated with A. Then for any vector (state) $v \in \mathcal{H}$ and any Borel set $Y \in \mathcal{B}$, the probability that a measurement of the observable will yield a value in the set Y, given the system is in state v, is taken to be $\nu_v(Y)$ (ref. 7, p. 260). Further, the expected value of this observable when the system is in state v is given by $\int_\Re x \, d\nu_v$ (ref. 7, p. 263). It remains to connect this approach to observables with our own. In the remainder of this discussion, the following assumption is understood.

Assumption. A is the self-adjoint operator associated with a given observable, and E: $\mathcal{B} \to Proj(\mathcal{H})$ is the spectral measure associated with A.

Note that the spectral measure E is an ortholattice homomorphism (ref. 7, p. 232), so the image of E is a Boolean subalgebra B of $Proj(\mathcal{H})$. As $Proj(\mathcal{H})$ is isomorphic to the questions $\mathfrak{Q}$ of the experimental system built from $\mathcal{H}$, there is a Boolean subalgebra B of $\mathfrak{Q}$ corresponding to the image of E. Then B is an observable quantity of the system. Let Z be the Stone space of B and define an extended real-valued map on Z by setting

$$f(F) = \inf\{\lambda \in \Re: E(-\infty, \lambda] \in F\} \tag{6.1}$$

Then by Proposition 5.7 the map f is measurable (with respect to the σ-algebra of subsets or Z generated by the clopen sets). In other words, f is a scaling of the observable quantity B. To complete the connection between the observable A and the observable quantity B, with scaling f, requires two technical results whose proofs are found in Appendix B (see also ref. 5, Chapter 5).

Lemma 6.1.6. For any state v and Borel set Y of the reals, $\nu_v(Y) = \mu_v(f^{-1}\, Y)$.

Lemma 6.1.7. For any state v, $\int_{\Re} x \, d\nu_v = \int_Z f \, d\mu_v$.

In Definition 5.5, $\mu_v(f^{-1}Y)$ is called the probability that a measurement of the observable quantity B with scaling f will yield a value in the Borel set Y, given the system is in state v. But by Lemma 6.1.6, this is exactly the value customarily associated with the probability of the observable A yielding a result in Y, given the system is in state v. Similarly, Lemma 6.1.7 shows that the notion of expected value described in Definition 5.5 agrees with that used in the usual Hilbert space model of quantum mechanics.

6.2. The Classical Model

In the classical model, propositions of the system correspond to certain subsets of the set T of states of the system. It is not assumed that each subset of T gives a proposition, but it is assumed that the propositions form a Boolean subalgebra B of the power set of T with intersection, union, and complementation giving the conjunction, disjunction, and negation of propositions. Given a state $t \in T$ and a proposition $A \subseteq T$, the probability of getting a *Yes* answer to proposition A when the system is in state t is either 0 or 1, depending on whether or not t belongs to the set A. Given such a classical system with state space T and propositions B, the aim is to produce an equivalent experimental system with probabilities.

Definition 6.2.1. Let E be the collection of all sequences $T_1, \ldots, T_n$ of pairwise disjoint members of B which cover T. Define an action of $\mathbb{O}$ on E by setting the ith term of the sequence $\sigma(T_1, \ldots, T_n)$ to be $\cup\{T_j: j \in \sigma(i)\}$.

The simple proof that E is a leveled set acted on by $\mathbb{O}$ is left to the reader.

Definition 6.2.2. Set $S = 2^T$, the collection of all maps from T into the two-element set $2 = \{0, 1\}$. For an experiment $e = (T_1, \ldots, T_n)$, define De to be the equivalence class of the decomposition $S \cong 2^{T_1} \times \ldots \times 2^{T_n}$. Further, let Pe: $S \to \{0, 1\}^n$ be the n-ary probability map whose ith component is defined by setting $(Pe)_i(\varphi) = 1$ if the support of φ is a singleton belonging to T_i, and 0 otherwise.

Proposition 6.2.3. The maps D: $E \to \mathfrak{D}(S)$ and P: $E \to \mathcal{P}(S)$ form an experimental system with probabilities. Further, $(A, A^c) \rightsquigarrow A$ is an isomorphism between the questions of the system and the Boolean algebra B, and the set T' of states of the system which are not null is isomorphic to T.

Proof. It is easy enough to see that the map D is an embedding, and clearly D satisfies Axiom 1. Axiom 2 follows as $2^{T_1 \cup T_2}$ is canonically isomorphic to $2^{T_1} \times 2^{T_2}$. For Axiom 3, suppose K is a finite set (T_i, T_i^c), $i = 1, \ldots, n$, of questions. Then any sequencing of the atoms of the Boolean subalgebra of B generated by $\{T_i: i \leq n\}$ yields an experiment from which each member of K can be built. Therefore Axiom 3 is satisfied, as every set of questions is compatible! That $(A, A^c) \rightsquigarrow A$ is an isomorphism follows directly from the definition of *and, or, not.*

That Pe is a probability map is easy to verify, and one notes that the states $\varphi: T \to 2$ which are not null are exactly the characteristic functions of singletons from T. Axiom 4 is obvious, and Axiom 5 follows easily from this description of null states. ∎

Informally, the notion of an experimental system created here is based on the superposition principle. To construct an experimental system that behaves classically, one must be rid of these superposed states. This is exactly what has been done by making them null states.

6.3. Other Models of Experimental Systems

Proposition 6.3.1. For any set S, the identical embedding from $\mathfrak{D}(S)$ to itself is an experimental system.

No proof is needed for this result, as it follows directly from the axioms. Note that for an experimental system D: $E \to \mathfrak{D}(S)$, the map D is an isomorphism between E and its range. Therefore every experimental system is equivalent to a subsystem of an experimental system $\mathfrak{D}(S)$. Roughly, the systems $\mathfrak{D}(S)$ play a role analogous to that of permutation groups in group theory. The experimental systems presented below will all literally be subsystems of $\mathfrak{D}(S)$.

Definition 6.3.2. An algebraic structure consists of a set S together with a family of operations on S. A decomposition $S \cong S_1 \times \ldots \times S_n$ is called a structural decomposition if each of the factors can be equipped with an algebraic structure making the structure S isomorphic to the product. Define Exp(S) to be the set of all equivalence classes of structural decompositions of S.

Note that if there exist algebraic structures on the factors S_i of a decomposition $S \cong S_1 \times \ldots \times S_n$, then they are uniquely determined, as the projections must be homomorphisms.

Proposition 6.3.3. For an algebraic structure S, the identical embedding of Exp(S) into $\mathfrak{D}(S)$ is an experimental system.

Proof. Obviously the identical embedding is an embedding which satisfies Axioms 1 and 2. Before verifying Axiom 3, recall a result from ref. 4, Theorem 5.8. There it is shown that that the set of binary structural decompositions form a subalgebra $\mathfrak{Q}'$ of the questions $\mathfrak{Q}$ of the system $\mathfrak{D}(S)$, and, moreover, that the Boolean subalgebras of $\mathfrak{Q}'$ are exactly the Boolean subalgebras of $\mathfrak{Q}$ which are contained in $\mathfrak{Q}'$.

To verify Axiom 3 it is enough to assume K is a finite set of questions with K compatible in $\mathfrak{D}(S)$. It then follows from the above remarks and Theorem 3.8 that K is contained in a finite Boolean subalgebra B of $\mathfrak{Q}'$ and that there is some n-ary decomposition d with $B = \{\sigma d\colon \|\sigma\| = 2\}$. Suppose $d = [S \cong S_1 \times \ldots \times S_n]$. As each decomposition $[S \cong S_1 \times \prod\{S_j\colon j \neq i\}]$ is in B, it is a structural decomposition. So the kernels of the projections are congruences, and this yields that $S \cong S_1 \times \ldots S_n$ is a structural decomposition. ■

Definition 6.3.4. A relational structure consists of a set S together with a nonempty binary relation on S. A decomposition $S \cong S_1 \times \ldots \times S_n$ is called a structural decomposition if the factors can be equipped with relations making the structure S isomorphic to the product. Define $\mathrm{Exp}(S)$ to be the set of all equivalence classes of structural decompositions.

Again, if there are relations on the factors of a decomposition $S \cong S_1 \times \ldots \times S_n$ making this a structural decomposition, then it is easy to see they are uniquely determined.

Proposition 6.3.5. For an relational structure S, the identical embedding of $\mathrm{Exp}(S)$ into $\mathfrak{D}(S)$ is an experimental system.

The proof of this result is substantially the same as the previous one, and again the crucial details are provided by ref. 4, Theorem 5.8. It is, however, a pleasant exercise to show for relational structures that each $[S \cong S_1 \times \prod\{S_j\colon j \neq i\}]$ being a structural decomposition implies $[S \cong S_1 \times \ldots \times S_n]$ is a structural decomposition.

Definition 6.3.6. A decomposition $S \cong S_1 \times \ldots \times S_n$ of a topological space S is called a structural decomposition if the factors can be equipped with topologies making the structure S isomorphic to the product. Define $\mathrm{Exp}(S)$ to be the set of all equivalence classes of structural decompositions.

As the projection operators associated with a product of topological spaces are both open and continuous, if there exist topologies on the factors of a decomposition $S \cong S_1 \times \ldots \times S_n$ making this a structural decomposition, then these topologies are uniquely determined.

Proposition 6.3.7. For a topological structure S, the identical embedding of $\mathrm{Exp}(S)$ into $\mathfrak{D}(S)$ is an experimental system.

Again the proof follows the above pattern, using ref. 4, Theorem 5.8. It is another nice exercise to show each $[S \cong S_i \times \prod\{S_j\colon j \neq i\}]$ being a structural decomposition of a topological space implies $[S \cong S_1 \times \ldots \times S_n]$

is a structural decomposition. The above results can also be combined in many ways. Rather than introduce further notation, a generic example is presented below.

Definition 6.3.8. A partially ordered topological group consists of a set S equipped with a group structure, a partial ordering, and a topology such that the group operations are continuous and $x \leq y$ implies $axb \leq ayb$ for all $a, b, x, y \in S$. A decomposition $S \cong S_1 \times \ldots \times S_n$ is said to be a structural decomposition if the factors S_i can be equipped with partially ordered topological group structures making the structure S isomorphic to the product. Define Exp(S) to be the set of all equivalence classes of structural decompositions.

Again, the structures on the factors realizing a structural decomposition are uniquely determined. The proof of the following result mirrors those above, using ref. 4, Theorem 5.13.

Proposition 6.3.9. For a partially ordered topological group S, the identical embedding of Exp(S) into $\mathfrak{D}(S)$ is an experimental system.

As a final comment, it is also shown in ref. 4 that for any algebraic or relational structure S, the observable quantities of the experimental systems Exp(S) are in complete correspondence with the Boolean sheaf representations of the structure S. See ref. 4, Propositions 6.5 and 6.7, for further details. No corresponding result is known for topological structures.

6.4. Other Models with Probabilities

Definition 6.4.1. Suppose $\eta: S \to [0, \infty)$ is such that $\eta(s) = 0$ for some $s \in S$. A decomposition $S \cong_\varphi S_1 \times \ldots \times S_n$ is called an η-decomposition if there exist maps $\eta_i: S_i \to [0, \infty)$ with $\eta(s_1, \ldots, s_n) \circ \varphi^{-1} = \Sigma\, \eta_i(s_i)$ for all $s_1, \ldots, s_n$.

For suggestive notation we assume that $0 \in S$ with $\eta(0) = 0$ and that $(0, \ldots, 0)$ is the element of the product $S_1 \times \ldots S_n$ corresponding to 0. Also, we abuse notation by using η in place of $\eta \circ \varphi^{-1}$. A simple observation is most useful.

Lemma 6.4.2. If $S \cong S_1 \times \ldots \times S_n$ is an η-decomposition, then the maps which realize this must be given by $\eta_i(s_i) = \eta(0, \ldots, s_i, \ldots, 0)$.

Proof. Suppose the maps η_i realize this decomposition being an η-decomposition. As each η_i is positive and $\eta(0, \ldots, 0) = 0$, it follows that $\eta_i(0) = 0$. Therefore $\eta(0, \ldots, s_i, \ldots, 0) = \eta_1(0) + \ldots + \eta_i(s_i) + \ldots + \eta_n(0) = \eta_i(s_i)$. ■

Of course, this lemma is independent of the particular element chosen to be 0. The following corollary will be most useful.

Corollary 6.4.3. A decomposition $S \cong S_1 \times \ldots \times S_n$ is an η-decomposition if and only if $\eta(s_1, \ldots, s_n) = \eta(s_1, 0, \ldots, 0) + \ldots + \eta(0, \ldots, 0, s_n)$ for each $s_1, \ldots, s_n$.

Before describing the systems built using the notion of η-decompositions, a technical lemma will be useful.

Lemma 6.4.4. Suppose $S \cong S_1 \times \ldots \times S_{2n}$ is a decomposition. If $S \cong (S_1 \times S_2) \times \ldots \times (S_{2n-1} \times S_{2n})$ and $S \cong (S_1 \times \ldots \times S_{2n-1}) \times (S_2 \times \ldots \times S_{2n})$ are η-decompositions, then so is $S \cong S_1 \times \ldots \times S_{2n}$.

Proof. Using the previous corollary, for each $s_1, \ldots, s_n$ we can express $\eta(s_1, \ldots, s_{2n})$ both as the sum $\eta(s_1, s_2, 0, \ldots, 0) + \ldots + \eta(0, \ldots, 0, s_{2n-1}, s_{2n})$ and as the sum $\eta(s_1, 0, s_3, 0, \ldots, s_{2n-1}, 0) + \eta(0, s_2, 0, s_4, \ldots, 0, s_{2n})$. Combining these, it follows that $\eta(s_1, \ldots, s_{2n})$ can be expressed as the sum $\eta(s_1, 0, \ldots, 0) + \ldots + \eta(0, \ldots, 0, s_{2n})$. The result then follows from the previous corollary. ■

Proposition 6.4.5. Suppose that D: $E \to \mathfrak{D}(S)$ is an experimental system and η: $S \to [0, \infty)$ with $\eta(s) = 0$ for some $s \in S$. Then the restriction of D to the set $E' = \{e$: De is an equivalence class of η-decompositions$\}$ is an experimental system.

Proof. Note that if d is an equivalence class of η-decompositions, then σd is also an equivalence class of η-decompositions. It follows that E' is closed under the action of $\mathbb{O}$, hence forms a leveled set acted on by $\mathbb{O}$. Clearly the restriction of D to E' is an embedding, and Axioms 1 and 2 are trivially satisfied. An auxiliary result is useful to establish Axiom 3.

Suppose e is an n-ary experiment in E' and f is a binary experiment in E'. If e, f can be built from a common experiment g, we claim there is an experiment h in E' which is built from g, and from which each of e, f can be built. Indeed, if e, f can be built from g, then $e = \sigma g$ and $f = \phi g$ for some ordered partitions σ, ϕ. Set

$$h = (\sigma(1) \cap \phi(1), \sigma(1) \cap \phi(2), \ldots, \sigma(n) \cap \phi(2))g$$

Clearly h is built from g, and one calculates $e = (\{1, 2\}, \{3, 4\}, \ldots, \{2n - 1, 2n\})h$ and $f = (\{1, 3, \ldots, 2n - 1\}, \{2, 4, \ldots, 2n\})h$. That h is an element of E' then follows directly from the previous lemma.

To verify Axiom 3 it is enough to assume $K = (k_i)$, $i = 1, \ldots, n$, is a finite set of questions in E' with $D[K]$ compatible. Then as E is an experimental system, there is some $g \in E$ from which each member of K can be built. Apply the above remarks to k_1, k_2 to obtain an experiment h_1 in E' which is built from g and from which each of k_1, k_2 can be built. As each of h_1, k_3 can be built from g, apply the above remarks again to h_1, k_3 to obtain an experiment h_2 in E' which is built from g and from which each of h_1, k_3 can be built. Then clearly each of k_1, k_2, k_3 can be built from h_2. Proceeding in this fashion shows there is an experiment in E' from which each member of K can be built, establishing Axiom 3. ■

Definition 6.4.6. Let e be an experiment in E' with $De = [S \cong S_1 \times \ldots \times S_n]$. Define a map Pe: $S \to [0, 1]^n$ as follows. For $s \in S$, let the corresponding element of the product be $(s_1, \ldots, s_n)$. Then the ith component of Pe is defined by setting $(Pe)_i(s)$ to be $\eta(0, \ldots, s_i, \ldots, 0)/\eta(s_1, \ldots, s_n)$ if the denominator is nonzero, and 0 otherwise.

One must check that this definition is independent of the particular choice of element from the equivalence class De, but this is straightforward. The following result provides our experimental systems with probabilities.

Proposition 6.4.7. The maps D: $E' \to \mathcal{D}(S)$ and P: $E' \to \mathcal{P}(S)$ form an experimental system with probabilities. Further, a state s of this system is null if and only if $\eta(s) = 0$.

Proof. For each e in E', the decomposition De is an η-decomposition. It follows from the above corollary that Pe is a probability map. Axiom 4 is trivial, and Axiom 5 follows easily. ■

Proposition 6.4.8. Let $(S, \cdot)$ be a real, or complex, inner product space and set $\eta(s) = \|s\|^2$. Let E be the set of all vector space structure preserving decompositions of S, and E' be all those members of E which are η-decompositions as well. Then E' is an experimental system with probabilities whose questions $\mathcal{Q}$ are isomorphic to the orthomodular poset of splitting subspaces of S.

Proof. The results of Section 6.3 show E is an experimental system. Its questions are easily seen to correspond to the set of all ordered pairs (A_1, A_2) of subspaces of S which are disjoint and together span S. But the question corresponding to (A_1, A_2) belongs to E' if and only if $\|s_1\|^2 + \|s_2\|^2 = \|s_1 + s_2\|^2$ for all $s_1 \in A_1$ and $s_2 \in A_2$. This is equivalent to A_1 and A_2 being orthogonal. ■

A final comment. If the inner product space in the previous result is a Hilbert space $\mathcal{H}$, then the experimental system with probabilities created corresponds exactly to that of Section 6.1.

APPENDIX A

Proofs of several technical results are presented in this appendix. For proofs involving equivalence classes of decompositions $[S \cong_f S_1 \times \ldots \times$

S_n] it will be most convenient to first show that each equivalence class has a canonical representative, then base calculations on the canonical representatives of the classes involved. Before developing this theory of canonical representatives, some notation is introduced and a few basic facts reviewed.

Definition A.1. For an equivalence relation θ on S and $s \in S$, let s/θ denote the equivalence class of θ containing s, and S/θ be the set of all equivalence classes of θ.

Definition A.2. Given equivalence relations θ, ϕ on S, define their relational product $\theta \circ \phi$ to be $\{(a, c)$: exists, $b \in S$ with $(a, b) \in \theta$ and $(b, c) \in \phi\}$. If $\theta \circ \phi = \phi \circ \theta$, then θ, ϕ are said to permute. It is easy to see that the relational product of two permuting equivalence relations is again an equivalence relation.

The following result is well known and easy to verify.

Proposition A.3. The collection $Eq(S)$ of all equivalence relations on a set S forms a complete lattice under set inclusion. Meets in this lattice are given by intersection. The join of two permuting equivalence relations is given by their relational product.

The following notion is of fundamental importance. While studied in other papers, the choice of name comes from ref. 4.

Definition A.4. A Boolean subsystem of $Eq(S)$ is a Boolean sublattice of $Eq(S)$ consisting of pairwise permuting elements. From the above proposition, finite meets in a Boolean subsystem are given by intersection, and finite joins by relational product.

Finite Boolean subsystems of $Eq(S)$ are closely related to direct product decompositions of S. The exact relationship is provided by the following lemma.

Lemma A.5. Given a sequence $\theta_1, \ldots, \theta_n$ of equivalence relations on S, define a map $\varphi: S \to S/\theta_1 \times \ldots \times S/\theta_n$ by $\varphi(s) = (s/\theta_1, \ldots, s/\theta_n)$. These are equivalent.

(1) $\varphi: S \to S/\theta_1 \times \ldots \times S/\theta_n$ is an isomorphism.
(2) The members of the sequence $\theta_1, \ldots, \theta_n$ which are not the largest relation on S are distinct and comprise the coatoms of a finite Boolean subsystem of $Eq(S)$.

Note that if θ_i is the largest relation on S, then S/θ_i is a one-element set and has no essential effect on the product.

A proof of the above lemma is found in ref. 4, Proposition 2.3.4. This result is the key to providing canonical representatives for equivalence classes of decompositions.

Lemma A.6. Let $S \cong_f S_1 \times \ldots \times S_n$. Then there is exactly one sequence $\theta_1, \ldots, \theta_n$ of equivalence relations with $[S \cong_f S_1 \times \ldots \times S_n] = [S \cong_\varphi S/\theta_1 \times \ldots \times S/\theta_n]$, where φ is the natural map given by $\varphi(s) = (s/\theta_1, \ldots, s/\theta_n)$.

Proof. Define $\theta_i = \ker f_i$. Then the maps $h_i: S_i \to S/\theta_i$ defined by setting $h_i(x) = f_i^{-1}(x)$ are isomorphisms and $h_i \circ f_i = \varphi_i$. It follows that φ is an isomorphism and that $S \cong_f S_1 \times \ldots \times S_n$ is equivalent to $S \cong_\varphi S/\theta_1 \times \ldots \times S/\theta_n$. If $S \cong_f S_1 \times \ldots \times S_n$ is equivalent to $S \cong_\chi S/\phi_1 \times \ldots \times S/\phi_n$, where $\chi(s) = (s/\phi_1, \ldots, s/\phi_n)$, then the decompositions $S \cong_\varphi S/\theta_1 \times \ldots \times S/\theta_n$ and $S \cong_\chi S/\phi_1 \times \ldots \times S/\phi_n$ are equivalent. Let $g_i: S/\theta_i \to S/\phi_i$ be the isomorphism with $g_i \circ \varphi_i = \chi_i$ realizing this equivalence. Then $\phi_i = \ker \chi_i = \ker g_i \circ \varphi_i = \ker \varphi_i = \theta_i$. ■

Having canonical representatives of equivalence classes of decompositions, it remains to describe the action of $\mathcal{O}$ on $\mathcal{D}(S)$ in terms of these canonical representatives. This is provided by the following lemma, whose proof follows from ref. 4, Corollary 2.3.5.

Lemma A.7. For a decomposition $S \cong S/\theta_1 \times \ldots \times S/\theta_n$ and an ordered partition σ, $\sigma[S \cong S/\theta_1 \times \ldots \times S/\theta_n] = [S \cong S/\phi_1 \times \ldots \times S/\phi_k]$, where $\phi_i = \cap \{\theta_j: j \in \sigma(i)\}$.

A simple example may be of benefit. If $S \cong S/\theta_1 \times \ldots \times S/\theta_4$ is a decomposition and $\sigma = (\{2\}, \{1,4\}, \{3\})$, then

$$\sigma[S \cong S/\theta_1 \times \ldots \times S/\theta_4] = [S \cong S/\theta_2 \times S/(\theta_1 \cap \theta_4) \times S/\theta_3]$$

This method to calculate the action of $\mathcal{O}$ is used advantageously in the technical proofs of results from Section 3. The proof of Lemma 3.2 is the first task. For the convenience of the reader, this lemma is restated below.

Lemma 3.2. If e, f are compatible questions, then there is a unique experiment g with four outcomes such that $e = (\{1, 2\}, \{3, 4\})g$ and $f = (\{1, 3\}, \{2, 4\})g$. The experiment g is called the standard refinement of the ordered pair (e, f).

Proof. As e, f can be conducted simultaneously, there is an n-ary experiment h and ordered partitions σ, ϕ of n with $e = \sigma h$ and $f = \phi h$. Set

$$\mu = (\sigma(1) \cap \phi(1), \sigma(1) \cap \phi(2), \sigma(2) \cap \phi(1), \sigma(2) \cap \phi(2))$$

As $\phi(1)$, $\phi(2)$ partition n, $\mu(1) \cup \mu(2) = \sigma(1)$. Using this and other, similar equations, it follows from Definition 2.3 that $(\{1, 2\}, \{3, 4\})\mu = \sigma$ and $(\{1, 3\}, \{2, 4\})\mu = \phi$. Therefore $g = \mu h$ satisfies the above conditions.

It remains to show that there is only one such experiment. Suppose g and g' satisfy the above conditions, and suppose $Dg = [S \cong S/\theta_1 \times \ldots \times S/\theta_4]$ and $Dg' = [S \cong S/\chi_1 \times \ldots \times S/\chi_4]$. By Lemma A.5, the members of $\theta_1, \ldots, \theta_4$ which differ from the largest relation are distinct and comprise exactly the coatoms of a finite Boolean subsystem of $Eq(S)$. It follows that

$$\theta_1 = (\theta_1 \cap \theta_2) \vee (\theta_1 \cap \theta_3) \qquad \text{and} \qquad \chi_1 = (\chi_1 \cap \chi_2) \vee (\chi_1 \cap \chi_3)$$

Here $\vee$ represents the join in the lattice of equivalence relations. As $(\{1, 2\}, \{3, 4\})g = (\{1, 2\}, \{3, 4\})g'$ and $(\{1, 3\}, \{2, 4\})g = (\{1, 3\}, \{2, 4\})g'$, it follows from Lemmas A.6 and A.7 that $\theta_1 \cap \theta_2 = \chi_1 \cap \chi_2$ and $\theta_1 \cap \theta_3 = \chi_1 \cap \chi_3$. Thus $\theta_1 = \chi_1$. In a similar fashion, one shows $\theta_i = \chi_i$ for $i \leq 4$, hence $Dg = Dg'$, and as D is an embedding, $g = g'$. ■

Definition A.8. For a set S, define *Fact S* to be the set of all ordered pairs (θ, θ') of equivalence relations on S for which S is canonically isomorphic to the product $S/\theta \times S/\theta'$. Define a unary operation $\perp$ on *Fact S* by setting $(\theta, \theta')^{\perp} = (\theta', \theta)$ and a relation $\leq$ by setting $(\theta, \theta') \leq (\phi, \phi')$ if all equivalence relations involved are in a Boolean subsystem of $Eq(S)$ and $\theta \subseteq \phi$. Constants 0 and 1 are defined to be the two ordered pairs consisting of the smallest and largest equivalence relations on S.

A proof of the following result is found in ref. 3, Theorem 3.5.

Theorem A.9. (*Fact S*, $\leq$, $\perp$, 0, 1) is an orthomodular poset.

The next task is a proof of the main result of Section 3, Theorem 3.8. This proof is somewhat involved and will be broken into several pieces based on results from refs. 3 and 4. Throughout, the following assumption will be understood.

Assumption. Assume $\mathfrak{Q}$ is the set of questions of an experimental system based on decompositions of the set S and φ is the map from $\mathfrak{Q}$ to *Fact S* defined by setting $\varphi e = (\theta', \theta)$ if $De = [S \cong S/\theta \times S/\theta']$. Note that the order of the equivalence relations is reversed by this map.

Lemma A.10. Let K be a subset of $\mathfrak{Q}$. Then K is compatible if and only if $\varphi[K]$ is contained in a Boolean subalgebra of *Fact S*.

Proof. Note that K is compatible iff each finite subset of K is compatible, and $\varphi[K]$ is contained in a Boolean subalgebra of *Fact S* iff each finite subset of $\varphi[K]$ is contained in a Boolean subalgebra of *Fact S*. Therefore it is enough to establish the result under the assumption that K is finite. So assume K is a finite subset of $\mathfrak{Q}$.

Recall a fact proved in ref. 4, Corollary 3.6. A finite subset (θ_i, θ_i'), $i = 1, \ldots, m$, is contained in a Boolean subalgebra of *Fact S* iff $\{\theta_i, \theta_i' : i \leq m\}$ is contained in a Boolean subsystem of $Eq(S)$.

Suppose K is compatible. By definition, there exists an n-ary experiment g from which each member of K is built. Suppose that $Dg = [S \cong S/\alpha_1 \times \ldots \times S/\alpha_n]$. Then by Lemma A.5, the relations $\alpha_1, \ldots, \alpha_n$ lie in a Boolean subsystem of *Fact S*. But by Lemma A.7, each of the relations occurring in the decomposition Dk of a member of K will also belong to this Boolean subsystem. By the above-mentioned result, it follows that $\varphi[K]$ is contained in a Boolean subalgebra of *Fact S*.

Conversely, assume $\varphi[K]$ is contained in a Boolean subalgebra of *Fact S*. As finitely generated Boolean algebras are finite, $\varphi[K]$ is in some finite Boolean subalgebra of *Fact S*. Let $\theta_1, \ldots, \theta_n$ be the coatoms of this Boolean algebra. Then by Lemma A.5, $S \cong S/\theta_1 \times \ldots \times S/\theta_n$. Let d be the equivalence class of this decomposition. Suppose $k \in K$ and $Dk = (\alpha, \alpha')$. Setting $A = \{i: \alpha \leq \theta_i\}$ and $A' = \{i: \alpha' \leq \theta_i\}$ it follows from Lemma A.7 that $Dk = \sigma g$ for $\sigma = (A, A')$. So by Axiom 3, K is compatible. ■

Lemma A.11. For e, f compatible questions, $\varphi(e \text{ or } f)$ and $\varphi(e \textit{ and } f)$ are the join and meet of φe, φf in *Fact S*.

Proof. Recall facts proved in ref. 4, Lemmas 3.1, 3.2. If (ϕ_1, ϕ_1') and (ϕ_2, ϕ_2') belong to a Boolean subalgebra of *Fact S*, then the join of these elements in *Fact S* is given by $(\phi_1 \circ \phi_2, \phi_1' \cap \phi_2')$ and the meet of these elements is given by $(\phi_1 \cap \phi_2, \phi_1' \circ \phi_2')$.

Assume that e, f are compatible questions. Let g be their standard refinement and assume $Dg = [S \cong S/\theta_1 \times \ldots \times S/\theta_4]$. From the definition of standard refinements it follows that $\varphi e = (\theta_3 \cap \theta_4, \theta_1 \cap \theta_2)$ and $\varphi f = (\theta_2 \cap \theta_4, \theta_1 \cap \theta_3)$. Similarly, using the definition of the logical operation, one obtains $\varphi(e \textit{ or } f) = (\theta_4, \theta_1 \cap \theta_2 \cap \theta_3)$ and $\varphi(e \textit{ and } f) = (\theta_2 \cap \theta_3 \cap \theta_4, \theta_1)$. As e, f are assumed compatible, it follows from the previous lemma that φe, φf belong to a Boolean subalgebra of *Fact S*. Using the above-mentioned result for computing joins and some simple Boolean arithmetic, it follows that the join of φe and φf is given by $\varphi(e \textit{ or } f)$, with a similar result for meets. ■

Lemma A.12. For questions e and f, e *implies* f if and only if $\varphi e \leq \varphi f$.

Proof. By definition, e *implies* f means e, f are compatible and $((\textit{not } e) \textit{ or } f) = \textit{True}$. Also, as *Fact S* is an orthomodular poset, $\varphi e \leq \varphi f$ is equivalent to φe, φf belonging to a Boolean subalgebra of *Fact S* and $(\varphi e)^{\perp} \vee \varphi f = 1$. Here $\vee$ denotes join in *Fact S*.

Lemma A.10 establishes that e, f are compatible iff φe, φf lie in a Boolean subalgebra of *Fact S*. To establish the result, it remains to show that ((*not* e) *or* f) $=$ *True* is equivalent to $(\varphi e)^{\perp} + \varphi f = 1$, under the assumption that e, f are compatible. With the trivial observations that $\varphi(\textit{not } e) = (\varphi e)^{\perp}$ and $\varphi(\textit{True}) = 1$, this follows easily by applying Lemma A.11 to φ((*not* e) *or* f). ■

A proof of the main result from Section 3, Theorem 3.8, can now be given. For the convenience of the reader, this theorem is restated below.

Theorem 3.8. (1) ($\mathfrak{Q}$, *implies, not, False, True*) is an orthomodular poset. (2) A set K of questions is contained in a Boolean subalgebra of $\mathfrak{Q}$ iff K is compatible, which occurs iff any two questions in K are compatible. (3) The operations in any Boolean subalgebra of $\mathfrak{Q}$ are given by *and, or, not, False, True*. (4) If B is a finite Boolean subalgebra of $\mathfrak{Q}$ with n atoms, then there exists an n-ary experiment g such that $B = \{\sigma g\colon \|\sigma\| = 2\}$.

Proof. Recall a few facts. An orthomodular poset P is called regular if a necessary and sufficient condition for a subset $A \subseteq P$ to be contained in a Boolean subalgebra of P is that each pair of elements of A is contained in a Boolean subalgebra of P. In ref. 4, Theorem 4.9 it is shown that *Fact S* is regular. A subalgebra S of an orthomodular poset P is called a compatible subalgebra if two elements of S are in a Boolean subalgebra of S if and only if they are in a Boolean subalgebra of P. Suppose S is a subset of a regular orthomodular poset P which is closed under orthocomplementation. If every $x, y \in S$ which are in a Boolean subalgebra of P have their meet in P belonging to S, then by ref. 4, Corollary 2.1.14, S is a compatible subalgebra of P which itself is a regular orthomodular poset.

Consider the subset $\varphi[\mathfrak{Q}]$ of *Fact S*. Observing that $(\varphi e)^{\perp} = \varphi$ (*not* e), it follows that $\varphi[\mathfrak{Q}]$ is closed under orthocomplementation. But if φe, φf are contained in a Boolean subalgebra of *Fact S*, then by Lemma A.10, e, f are compatible. By Lemma A.11, the meet of φe, φf in *Fact S* is φ(e *and* f), which clearly belongs to $\varphi[\mathfrak{Q}]$. By the above-mentioned results, $\varphi[\mathfrak{Q}]$ is a compatible subalgebra of *Fact S* which itself is a regular orthomodular poset.

Note that $\varphi e = \varphi f$ implies that $De = Df$, and as D is an embedding, this implies $e = f$. Using Lemma A.12 and the observation that $\varphi(\textit{not } e) = (\varphi e)^{\perp}$, it follows that φ is an isomorphism between the structure ($\mathfrak{Q}$, *implies, not, False, True*) and the structure $(\varphi[\mathfrak{Q}], \leq, \perp, 0, 1)$. Therefore ($\mathfrak{Q}$, *implies, not, False, True*) is also a regular orthomodular poset. This establishes (1).

As $\varphi[\mathfrak{Q}]$ is a regular orthomodular poset which is a compatible subalgebra of the regular orthomodular poset *Fact S*, it follows that a subset of $\varphi[\mathfrak{Q}]$ is contained in a Boolean subalgebra of $\varphi[\mathfrak{Q}]$ if and only if it is contained in a Boolean subalgebra of *Fact S*. Lemma A.10 then shows that a subset K of $\mathfrak{Q}$ is contained in a Boolean subalgebra of $\mathfrak{Q}$ if and only if K is compatible. The remainder of the equivalence stated in (2) follows from what has already been shown and the regularity of $\mathfrak{Q}$.

If e, f are elements of a Boolean subalgebra B of $\mathfrak{Q}$, then by Lemma A.11, φ (e *or* f) and φ(e *and* f) are the join and meet of φe, φf in *Fact S*. As these elements belong to $\varphi[\mathfrak{Q}]$, they must also be the join and meet of φe, φf in $\varphi[\mathfrak{Q}]$. Since $\varphi[B]$ is a Boolean subalgebra of $\varphi[\mathfrak{Q}]$, it follows from Lemma 3.7 that φ(e *or* f) and φ(e *and* f) are the join and meet of φe, φf in B. As φ is an isomorphism, joins and meets in B are given by *or*, *and*. As orthocomplementation on $\mathfrak{Q}$ is given by *not*, this must also be the orthocomplementation on any subalgebra of $\mathfrak{Q}$, and clearly any subalgebra inherits the bounds *False, True* of $\mathfrak{Q}$. This establishes (3).

Let B be a finite Boolean subalgebra of $\mathfrak{Q}$ with n atoms $e_1, \ldots, e_n$. By part (2), B can be conducted simultaneously, so there is an m-ary experiment h with $B \subseteq \{\sigma h: \|\sigma\| = 2\}$. Let σ_i be such that $e_i = \sigma_i h$, and let $X_i \subseteq \{1, \ldots, m\}$ be such that $\sigma_i = (X_i, X_i^c)$. Using the fact that operations in B are given by *and, or, not*, it follows from Lemma 3.3 that $(X_1, \ldots, X_n)$ is an ordered partition of m. Setting $g = (X_1, \ldots, X_n)h$, one can verify $B = \{\sigma g{:}\|\sigma\| = 2\}$. This establishes (4). ■

To conclude the technical proofs of results from Section 3, it remains only to discuss Propositions 3.9–3.11. For the proof of Proposition 3.9, one first notes that the weaker version of Axiom 3 can be used to prove a weaker version of Lemma A.10 applying only to sets K consisting of two binary questions. Then the proof of Theorem 3.8 can be nearly duplicated to prove Proposition 3.9. The only small change is in establishing the second condition of proposition 3.9, which actually becomes slightly easier.

Only an example is required to establish Proposition 3.10. Take any set S and let E be the collection of all equivalence classes of decompositions $[S \cong S_1 \times \ldots \times S_n]$ for which at most four of the factors have more than one element. Clearly E is a subset of $\mathfrak{D}(S)$ which is closed under the action of $\mathbb{O}$, hence is a leveled set acted on by $\mathbb{O}$. Define D: $E \to \mathfrak{D}(S)$ to be the identical embedding. Axioms 1 and 2 are trivially satisfied. To verify the weakened version of Axiom 3, one must show that any two binary decompositions which have a common refinement in $\mathfrak{D}(S)$ have a common refinement in E. But the standard refinement will have only four factors, hence surely be a member of E. To show this system does not satisfy Axiom 3, take any decomposition d of S with at least five factors having more than one element. Let K be the set of all binary decompositions which can be built from d. Surely K is compatible in $\mathfrak{D}(S)$, but one cannot build all members of K from a single member of E.

The proof of Proposition 3.11 will conclude our work on Section 3. For the convenience of the reader, the result is restated below.

Proposition 3.11. For an experimental system, Axiom 3 holds for any set K of experiments, binary or otherwise.

Proof. Note that it is sufficient to prove Axiom 3 holds for any finite set K of experiments. Let K be a finite set of experiments with $D[K]$ compatible. Then for $K' = \{\sigma k: k \in K, \|\sigma\| = 2\}$, it follows that $D[K']$ is compatible. So by Axiom 3, K' is compatible. Let g be an experiment from which each member of K' can be built. Suppose $Dg = [S \cong S/\theta_1 \times \dots \times S/\theta_n]$, and note that the experiment g may be assumed to be chosen so that none of the relations θ_i are the largest relation on S. By Lemma 7.5, $\theta_1, \dots, \theta_n$ are distinct and comprise exactly the coatoms of a finite Boolean subsystem of $Eq(S)$.

As each member of K' can be built from g, it follows from Lemma A.7 that any equivalence relation occurring in a decomposition of some member of K' must belong to this Boolean subsystem. Suppose $k \in K$ and that $Dk = [S \cong S/\phi_1 \times \dots \times S/\phi_m]$. As the decomposition of $(\{i\}, \{i\}^c)k$ is given by $[S \cong S/\phi_i \times S/\cap\{\phi_j: j \neq i\}]$, it follows that each ϕ_i belongs to this Boolean subsystem as well. Set $A_i = \{j: \phi_i \leq \theta_j\}$. Then $\sigma = (A_1, \dots, A_m)$ is an ordered partition of n, and by Lemma A.7, it follows that $Dk = \sigma(Dg)$. Therefore $k = \sigma g$, showing each member of K can be built from g. ■

APPENDIX B

The focus now turns to verifying the technical results from Section 6.

Lemma B.1. Let $P_1, P_2, P_1^\perp, P_2^\perp$ be projection operators of a Hilbert space. If the kernels of these projection operators lie in a common Boolean subsystem of $Eq(\mathcal{H})$, then these projections commute.

Proof. By symmetry, it is enough to show that P_1 and P_2 commute. Suppose v is in the range of P_1. Then $P_1^\perp v = 0$, so $v\,\theta_1^\perp\,0$. As $\theta_1^\perp = (\theta_1^\perp \cap \theta_2) \circ (\theta_1^\perp \cap \theta_2^\perp)$, there is some vector w with $v(\theta_1^\perp \cap \theta_2)w$ and $w(\theta_1^\perp)0$. It follows that w is in the ranges of both P_1 and P_2, and that $P_2 v = P_2 w$. So $P_1 P_2 v = P_1 P_2 w = P_2 w = P_2 v = P_2 P_1 v$. Similarly, if v is in the range of $P_1^\perp$, there is some w with $v(\theta_1 \cap \theta_2)w$ and $w(\theta_1 \cap \theta_2^\perp)0$. Then w is in the ranges of both $P_1^\perp$ and P_2, and $P_2 v = P_2 w$. Therefore $P_1 P_2 v = P_1 P_2 w = 0 = P_2 P_1 v$. As any vector can be expressed as a sum of vectors from the ranges of P_1 and $P_1^\perp$, the result follows. ■

The next task is to prove Proposition 6.1.3, which is restated below.

Proposition 6.1.3. The maps $D: E \to \mathcal{D}(\mathcal{H})$ and $P: E \to \mathcal{P}(\mathcal{H})$ form an experimental system with probabilities. Further, $(P, P^\perp) \rightsquigarrow P$ is an isomorphism between the orthomodular poset $\mathcal{Q}$ of questions of the system and $\mathrm{Proj}(\mathcal{H})$.

Proof. First we show D is an embedding which satisfies Axioms 1–3. Suppose $e = (P_1, \dots, P_n)$ is an experiment and $\mathcal{H} \cong \mathcal{H}/\theta_1 \times \dots \times \mathcal{H}/\theta_n$

is the canonical representative of the equivalence class De. Then the relations θ_i are the kernels of the projections P_i, so D is an embedding. Clearly if e is n-ary, then so is De, so D satisfies Axiom 1. As the range of the projection $\Sigma_i^k P_i$ is canonically isomorphic to the product of the ranges of the summands, it follows that D satisfies Axiom 2.

To verify Axiom 3, suppose K is a set of questions with $D[K]$ compatible. By the nature of the axiom, there is no loss of generality to assume that K is finite. Let $(P_i, P_i^\perp)$ for $i = 1, \ldots, m$ be an enumeration of K, with $\theta_i = \ker P_i$ and $\theta_i^\perp = \ker p_i^\perp$. As K is compatible, Lemmas A.5 and A.7 show that $\{\theta_i, \theta_i^\perp: i \leq m\}$ is contained in a Boolean subsystem of $Eq(\mathcal{H})$. Using the previous lemma, the projections P_i, $P_i^\perp$ for $i = 1, \ldots, m$ all commute, hence are contained in a finite Boolean subalgebra of $Proj(\mathcal{H})$. For $R_1, \ldots, R_n$ the atoms of this Boolean algebra, it follows that $(R_1, \ldots, R_n)$ is an experiment from which each member of K can be built. This establishes Axiom 3.

It is easy to establish Axioms 4 and 5, hence D and P form an experimental system with probabilities. For the further comment, the map $(P, P^\perp) \rightsquigarrow P$ is surely a bijection which preserves orthocomplementation. Suppose that $e_1 = (P_1, P_1^\perp)$ and $e_2 = (P_2, P_2^\perp)$ are questions. By Definition 3.4, e_1 *implies* e_2 iff there is a ternary experiment g with $e_1 = (\{1\}, \{2,3\})g$ and $e_2 = (\{1,2\},\{3\})g$. So if e_1 *implies* e_2, then $P_1 \leq P_2$. Conversely, if $P_1 \leq P_2$, then $(P_1, P_2 - P_1, P_2^\perp)$ is an experiment realizing e_1 *implies* e_2. Therefore this map is an order isomorphism as well, hence an isomorphism of orthomodular posets. ■

Our attention now turns to Lemmas 6.1.6 and 6.1.7, which relate self-adjoint operators of a Hilbert space to observable quantities and scalings of an experimental system. In this discussion, the following assumption is understood.

Assumption. A is a self-adjoint operator with spectral measure E. B is the Boolean subalgebra of $\mathfrak{Q}$ corresponding to the image of E. Z is the Stone space of B and $\mathfrak{D}$ is the σ-algebra generated by the clopen sets of Z. For each nonzero vector v, ν_v is the probability measure on the Borel sets $\mathcal{B}$ of the reals defined in Proposition 6.1.5, and μ_v is the probability measure on $\mathfrak{D}$ defined in Proposition 5.4. Finally, f is the extended real-valued map on Z defined by $f(F) = \inf\{\lambda \in \mathfrak{R}: E(-\infty, \lambda] \in F\}$.

For convenience, the distinction between elements of B, which are ordered pairs $(E(Y), E(Y)^\perp)$ of projections, and their corresponding elements $E(Y)$ of the image of E will be blurred. This practice begins in the following result.

Lemma B.2. For each Borel set Y of the reals, $\nu_v(Y) = \mu_v(E(Y)^*)$.

Proof. Note that $\mu_v(E(Y)^*)$ is an abuse of notation for $\mu_v((E(Y), E(Y)^{\perp})^*)$. By Proposition 5.4 and Definition 6.1.2 this has value $\|E(Y)v\|^2/\|v\|^2$. As $E(Y)$ is a projection, one verifies easily that $\|E(Y)v\|^2 = (v, E(Y)v)$. But by Proposition 6.1.5, $\nu_v(Y) = (v, E(Y)v/\|v\|^2$. ■

When working with measures and σ-algebras, the union of an increasing sequence of sets, or the intersection of a decreasing sequence of sets is referred to as the limit of the sequence. A fundamental property of measures is that they preserve such limits, i.e. $\mu(\lim A_n) = \lim \mu(A_n)$. Using this fact and the previous lemma, it is a simple matter to establish the following.

Lemma B.3. For any λ, $\mu_v(\lim E(-\infty, \lambda + 1/n]^*) = \nu_v(-\infty, \lambda]$. Further, $\mu_v(\lim E(-\infty, -n]^*) = 0$ and $\mu_v(\lim E(n, \infty)^*) = 0$.

The following is a simple consequence of the definition of $f(F)$. Its proof is left to the reader as well.

Lemma B.4. For any real number λ, $f(F) \leq \lambda$ iff $E(-\infty, \lambda + 1/n] \in F$ for each natural number n. Similarly, $f(F) = -\infty$ iff $E(-\infty, -n] \in F$ for each natural number n, and $f(F) = \infty$ iff $E(n, \infty) \in F$ for each natural number n.

With the aid of these preliminary results, the first of the two technical lemmas from Section 6.1 can be established.

Lemma 6.1.6. For any state v and Borel set Y of the reals, $\nu_v(Y) = \mu_v(f^{-1}Y)$.

Proof. Note first that the previous two lemmas give $\mu_v(f^{-1}\{-\infty, \infty\}) = 0$. Therefore, for any measurable subset X of the extended reals it follows that the measure of $f^{-1}X$ agrees with that of $f^{-1}(X\backslash\{\pm\infty\})$. This essentially removes the difficulties arising from f being an extended real-valued map and ν_v being a measure on the Borel subsets of the reals.

Consider the set $\Gamma = \{Y \in \mathcal{B}\colon f^{-1}\ Y \in \mathcal{D}$ and $\mu_v(f^{-1}Y) = \nu_v(Y)\}$. Clearly $\emptyset \in \Gamma$ and from the above discussion, $\mathfrak{R} \in \Gamma$ as well. For a real number λ, it follows from the above lemmas that $\mu_v(f^{-1}[-\infty, \lambda])$ is equal to $\nu_v(-\infty, \lambda]$. By the above discussion, the measure of $f^{-1}[-\infty, \lambda]$ agrees with that of $f^{-1}(-\infty, \lambda]$, so Γ contains the bounded chain $\mathcal{C}$ of subsets of the reals consisting of $\emptyset$, $\mathfrak{R}$, and the family $\{(-\infty, \lambda]\colon \lambda \in \mathfrak{R}\}$.

It is a simple matter to verify that Γ is closed under finite disjoint unions and the difference of sets $Y\backslash X$ with $X \subseteq Y$. It follows that Γ contains the Boolean algebra $\mathcal{B}_0$ of subsets of the reals generated by the bounded chain $\mathcal{C}$. If we can show that Γ is a monotone class it will follow from ref. 6, Proposition I.4.2, that Γ contains the σ-algebra of subsets of the reals generated by $\mathcal{B}_0$, which is the whole of the Borel sets $\mathcal{B}$. This will establish the result.

Suppose Y_n is an increasing sequence of subsets of the reals with each $Y_n \in \Gamma$, and set $Y = \lim Y_n$. Clearly $Y \in \mathcal{B}$ and as $f^{-1}\ Y = \lim f^{-1}Y_n$ it follows that $Y \in \mathfrak{D}$. It remains to show $\mu_v(f^{-1}Y) = \nu_v(Y)$. But this follows easily, as the measures μ_v and ν_v both preserve limits and $\mu_v(f^{-1}Y_n) = \nu_v(Y_n)$ by assumption. ■

The second, and final, technical result from Section 6.1 is established through a sequence of lemmas.

Lemma B.5. For $a < b$, $a \cdot \nu_v(a,b] \leq \int_{E(a,b]^*} f\, d\mu_v \leq b \cdot \nu_v\,(a,b]$.

Proof. By lemma B.2. $\mu_v(E(a,b]^*) = \nu_v(a,b]$. But if $F \in E(a,b]^*$, then $E(a,b] \in F$. This implies $E(-\infty,b] \in F$ and $E(-\infty,a] \notin F$. It follows that $a \leq f(F) \leq b$. ■

Lemma B.6. For $a < b$ we have $\int_{(a,b]} x\, d\nu_v = \int_{E(a,b]^*} f\, d\mu_v$.

Proof. Call a function $s(x)$ a special lower step function on $(a,b]$ if there are $a = p_0 < \ldots < p_{n+1} = b$ with $s = \Sigma_0^n\, p_i \cdot \chi_{(p_i,p_{i+1}]}$, and call $t(x)$ a special upper step function on $(a, b]$ if there are $a = q_0 < \ldots < q_{m+1} = b$ with $t = \Sigma_0^m\, q_{i+1} \cdot \chi_{(q_i,q_{i+1}]}$. For such s, t we claim

$$\int_{(a,b]} s\, d\nu_v \leq \int_{E(a,b]^*} f\, d\mu_v \leq \int_{(a,b]} t\, d\nu_v$$

Indeed, by Lemma B.5 and the fact that the $E\,(p_i,p_{i+1}]^*$ partition $E(a,b]^*$, we have

$$\int_{(a,b]} s\, d\nu_v = \sum_{i=0}^{n} p_i \cdot \nu_v(p_i,\, p_{i+1}] \leq \sum_{i=0}^{n} \int_{E(p_i,p_{i+1}]^*} f\, d\mu_v = \int_{E(a,b]^*} f\, d\mu_v$$

An obviously similar argument establishes the other inequality.

It is a simple matter to construct an increasing sequence s_n of special lower step functions on $(a, b]$ with $s_n(x) \to x$ for each $x \in (a, b]$, and similarly one can find a decreasing sequence t_n of special upper step functions on $(a, b]$ with $t_n(x) \to x$ for each $x \in (a,b]$. As $s_0 \leq s_n$, $t_n \leq t_0$, we may apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{n\to\infty} \int_{(a,b]} s_n\, d\nu_v = \int_{(a,b]} x\, d\nu_v = \lim_{n\to\infty} \int_{(a,b]} t_n\, d\nu_v$$

The result follows by the squeeze theorem. ■

The final lemma from Section 6.1 can now be established.

Lemma 6.1.7. For any state v, $\int_{\mathfrak{R}} x\, d\nu_v = \int_Z f\, d\mu_v$.

Proof. Let $X = \lim E(0, n]^*$. Note that an ultrafilter F will belong to $E(0, \infty)^* \setminus X$ if and only if $E(n, \infty) \in F$ for each natural number n. Therefore

$E(0, \infty)^* \setminus X$ equals $\lim E(n, \infty)^*$, hence by Lemma B.3 has measure zero. In a similar fashion, set $Y = \lim E(-n, 0]^*$. Then an ultrafilter F belongs to $E(-\infty, 0]^* \setminus Y$ if and only if $E(-\infty, -n] \in F$ for each natural number n. So $E(-\infty, 0] \setminus Y$ equals $\lim E(-\infty, n]^*$, and by Lemma A.3 has measure zero.

As f is positive on $E(0, \infty)^*$ and negative on $E(-\infty, 0]^*$, Lebesgue's monotone convergence theorem gives

$$\int_{E(0,n]^*} f\, d\mu_v \to \int_X f\, d\mu_v \qquad \text{and} \qquad \int_{E(-n,0]^*} f\, d\mu_v \to \int_Y f\, d\mu_v$$

But the function x is also positive on $(0, \infty)$ and negative on $(-\infty, 0]$,

$$\int_{(0,n]} x\, d\nu_v \to \int_{(0,\infty)} x\, d\nu_v \qquad \text{and} \qquad \int_{(-n,0]} x\, d\nu_v \to \int_{(-\infty,0]} x\, d\nu_v$$

Using the previous lemma and the fact that X and Y differ from $E(0, \infty)^*$ and $E(-\infty,0]^*$ by sets of measure zero yields the result. ■

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